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Discrete Approximation of Unbounded Operators and Approximation of their Spectra

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Let E be a Banach space over $\mathbb{C}$ and let the densely defined closed linear operator $A: \mathcal{D}(A) \subset E \to E$ be discretely approximated by the sequence $((A_n, \mathcal{D}(A_n)))_{n \in \mathbb{N}}$ of

A_n. Generalizing our own result, we show that $\sigma_a(A) \subset \liminf \sigma_\varepsilon(A_n) = \bigcup_{n \in \mathbb{N}} \bigcap_{k \geqslant n} \sigma_\varepsilon(A_k)$ holds for every $\varepsilon > 0$. We deduce that then for every compact set $K \subset \mathbb{C}$ $\lim_n \operatorname{dist}(\sigma_a(A) \cap K, \sigma_a(A_n)) = 0$ provided there exists $M > 0$ such that $\|(\lambda - A_n)^{-1}\| \leqslant M \operatorname{dist}(\lambda, \sigma(A_n))^{-1}$ holds for every n and every λ in the resolvent set $\rho(A_n)$ of A_n. We finally treat the problem under which conditions $\sigma_a(A)$ can be approximated from below. More precisely we investigate the problem: Under which assumptions does $\bigcap_{\varepsilon > 0} \bigcap_{n \in \mathbb{N}} \bigcup_{k \geqslant n} \sigma_{\varepsilon, a}(A_k) \subset \sigma_a(A)$ hold where $\sigma_{\varepsilon, a}(A)$ denotes the ε-approximate pseudospectrum?

Key Words: ε-pseudospectrum; discrete convergence; approximation of spectra.

1. INTRODUCTION

Let E be a Banach space and let $A: \mathcal{D}(A) \subset E \to E$ be a closed densely defined linear operator. By $\rho(A)$ we denote its resolvent set, by $\sigma(A)$ its spectrum, by $\sigma_p(A)$ the set of its eigenvalues, and finally by $\sigma_a(A)$ the *approximate point spectrum*

$$\sigma_a(A) = \{\lambda \in \sigma(A) : \inf\{\|(\lambda - A)\, x\| : \|x\| = 1, x \in \mathcal{D}(A)\} = 0\}.$$

Let E_0 be a core of A, i.e. a linear subspace of $\mathcal{D}(A)$ such that the closure $\overline{A|_{E_0}}$ of its restriction $A|_{E_0}$ equals A. Then

$$\lambda \in \sigma_a(A) \qquad \text{iff} \qquad \inf\{\|(\lambda - A)\, x\| : \|x\| = 1, x \in E_0\} = 0.$$

For $\varepsilon > 0$ let $\rho_\varepsilon(A)$ be the set $\{\lambda \in \rho(A) : \|(\lambda - A)^{-1}\| < \frac{1}{\varepsilon}\}$. Then $\mathbb{C} \setminus \rho_\varepsilon(A) = \sigma_\varepsilon(A)$ is called the *ε-pseudospectrum of A* (see [5, 13, 14]; notice that the

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ε-pseudospectrum as defined here differs from that one in [5] which is the open kernel of ours). The *distance* $\operatorname{dist}(K, L)$ from the bounded set K of the Banach space F to the nonempty set $L \subset F$ is defined by

$$\operatorname{dist}(K, L) = \sup_{z \in K} (\inf_{y \in L} \|z - y\|).$$

$\operatorname{dist}(K, L) = 0$ is obviously equivalent to $K \subset \bar{L}$. So we can define $\operatorname{dist}(\emptyset, L) = 0$ in a consistent manner.

It is well known that the mapping $T \to \sigma(T)$ from the set $\mathscr{L}(E)$ of all bounded linear operators on E into the set of all compact subsets of $\mathbb{C}$ equipped with the Hausdorff metric is not continuous. More precisely let (T_n) be a sequence in $\mathscr{L}(E)$ converging to $T \in \mathscr{L}(E)$ with respect to the operator norm. Then $\lim_n \operatorname{dist}(\sigma(T_n), \sigma(T)) = 0$ but $\lim_n \operatorname{dist}(\sigma(T), \sigma(T_n)) = 0$ does not hold in general (see [7, pp. 208–210]). However the following assertion is true.

PROPOSITION 1.1. *Let (T_n) be a sequence of bounded linear operators on the Banach space E which converges with respect to the operator norm to the operator T. Then to every pair $(\varepsilon_1, \varepsilon_2)$ of real numbers with $0 \leqslant \varepsilon_1 < \varepsilon_2$ there exists $n(\varepsilon_1, \varepsilon_2) \in \mathbb{N}$ such that $\sigma_{\varepsilon_1}(T) \subset \sigma_{\varepsilon_2}(T_n)$ holds for all $n \geqslant n(\varepsilon_1, \varepsilon_2)$.*

Proof. Let $n(\varepsilon_1, \varepsilon_2)$ be such that $\|T - T_n\| < \varepsilon_2 - \varepsilon_1$ holds for all $n \geqslant n(\varepsilon_1, \varepsilon_2)$. Assume that $\lambda \in \rho_{\varepsilon_2}(T_n)$ where $n \geqslant n(\varepsilon_1, \varepsilon_2)$ is fixed. Then

$$\|(T - T_n)(\lambda - T_n)^{-1}\| \leqslant \|T - T_n\| \, \|(\lambda - T_n)^{-1}\| < (\varepsilon_2 - \varepsilon_1)/\varepsilon_2 = 1 - \varepsilon_1/\varepsilon_2.$$

Therefore the series $\sum_{k=0}^{\infty} [(\lambda - T_n)^{-1} (T - T_n)]^k$ converges to $(I - (\lambda - T_n)^{-1} (T - T_n))^{-1}$. This in turn implies that

$$\lambda - T = (\lambda - T_n)(I - (\lambda - T_n)^{-1} (T - T_n))$$

is invertible and moreover

$$\begin{aligned} \|(\lambda - T)^{-1}\| &\leqslant \|(\lambda - T_n)^{-1}\| \, \|(I - (\lambda - T_n)^{-1} (T - T_n))^{-1}\| \\ &< \|(\lambda - T_n)^{-1}\| \cdot \frac{1}{1 - (1 - \varepsilon_1/\varepsilon_2)} \\ &\leqslant 1/\varepsilon_1 \end{aligned}$$

hence $\lambda \in \rho_{\varepsilon_1}(T)$. So $\rho_{\varepsilon_2}(T_n) \subset \rho_{\varepsilon_1}(T)$ for all $n \geqslant n(\varepsilon_1, \varepsilon_2)$ which proves the assertion. ∎

Remark 1.2. Harrabi [5] proved the following related theorem: If in addition to the hypotheses of Proposition 1.1 the norm of the resolvent is not constant on any open subset of the resolvent set then $\bigcap_{n \in \mathbb{N}} \bigcup_{k \geqslant n} \sigma_\varepsilon(T_k) = \bigcup_{n \in \mathbb{N}} \bigcap_{k \geqslant n} \sigma_\varepsilon(T_k) = \sigma_\varepsilon(T)$ holds for all $\varepsilon > 0$.

Our main aim is to show that the following assertion $\sigma_a(T) \subset \bigcup_{n\in\mathbb{N}} \bigcap_{k\geqslant n} \sigma_\varepsilon(T_k)$ which is slightly weaker than that one of Proposition 1.1 holds for all $\varepsilon > 0$ in a much more general situation. More precisely in the second section we shall prove a somewhat stronger formula in the case of discrete convergence which is motivated by numerical analysis and which we explain now:

Let E_1 be a dense linear subspace of the Banach space E, let (F_n) be a sequence of Banach spaces and for each n let $P_n : E_1 \to F_n$ be a not necessarily bounded linear mapping. If $\lim_n \|P_n x\|_n = \|x\|$ holds for every x in E_1 then $(E, E_1, (F_n), (P_n))$ is called a *discrete approximation scheme*. A sequence $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} F_n$ *converges discretely to* $x \in E_1$ if $\lim_{n\to\infty} \|x_n - P_n x\|_n = 0$ holds. Then we write $x = \mathrm{d}-\lim x_n$. Let now $(A, \mathscr{D}(A))$ be a closed densely defined linear operator on E and let $E_0 \subset E_1$ be a core of A such that $A(E_0) \subset E_1$ holds. For each n let $(A_n, \mathscr{D}(A_n))$ be a densely defined operator on F_n such that $P_n(E_1) \subset \mathscr{D}(A_n)$. We say that the sequence (A_n) *approximates A discretely* if for all $x \in E_0$ the sequence $(A_n P_n x)_n$ converges discretely to Ax, i. e. $\lim_n \|A_n P_n x - P_n A x\|_n = 0$ holds for all $x \in E_0$. Let us point out that this notion is weaker than the corresponding notion in [11, p. 168; 15, Sect. 1–2]. The notion of discrete approximation of operators in the literature which is closest to our one is to be found in [3, p. 368]. The main difference is that in our case the P_n need not be continuous. In [3, p. 366], however, there is also indicated how to overcome the restriction on P_n to be continuous made there.

EXAMPLES. (1) *Uniform convergence.* Let X be a Banach space, set $E = E_1 = \mathscr{L}(X)$, the Banach algebra of all bounded operators on X, set $F_n = E$ and $P_n = I$. For $T \in \mathscr{L}(X)$ we consider the multiplication operator $A = M_T : U \mapsto TU$. Then the sequence (T_n) converges to T with respect to the operator norm iff $(A_n) = (M_{T_n})$ approximates A discretely as defined above.

(2) *Pointwise (strong) convergence.* Here $E = E_1 = F_n$ and $P_n = I$ for all n. A sequence (A_n) of bounded operators A_n converges strongly to A iff (A_n) approximates A discretely.

(3) (Cf. [8].) Let E be a given Banach space and let (F_n) be an increasing sequence of closed linear subspaces with $F_\infty := \bigcup_n F_n$ dense in E. Moreover assume that each F_n is the range of a bounded projection P_n such that $\sup_n(\|P_n\|) < \infty$ as well as $P_{n+k}P_n = P_n$ for $k \geqslant 0$. Let $(A, \mathscr{D}(A))$ be a closed densely defined operator on E such that for all n $A|_{F_n} =: A_n$ maps $\mathscr{D}(A) \cap F_n =: \mathscr{D}(A_n)$ into F_n and moreover that $(A_n, \mathscr{D}(A_n))$ is densely defined and closable on F_n and finally that $\mathscr{D}(A) \cap F_\infty$ is a core of A. Then setting $E_1 = F_\infty$, $E_0 = \mathscr{D}(A) \cap F_\infty$ we obtain that the sequence (A_n) approximates A discretely. A similar setting is used in [4] within the context of approximation of ordinary differential operators on the real line.

(4) Let $E = L^2([0,1])$, $E_1 = \{f \in E : f$ continuous, $f(0) = f(1)\}$, $F_n = \mathbb{C}^n$ with the scalar product $(x \mid y) = \frac{1}{n} \sum_{k=1}^{n} \bar{x}_k y_k$, $P_n f = (f(\frac{1}{n}), \ldots, f(\frac{n}{n}))$, $E_0 = \{f \in E_1 : f' \in E_1\}$, and finally let $Af = f'$ with boundary condition $f(0) = f(1)$. For $A_n x = n(x_2 - x_1, \ldots, x_n - x_{n-1}, x_1 - x_n)$ the sequence (A_n) approximates A discretely.

(5) Same as (4) up to the A_n. Here we take $A_n x = n(x_2 - x_1, \ldots, x_n - x_{n-1}, 0)$.

(6) The spaces are the same as in (4). $Af = f' - f$ with boundary condition $f(0) = f(1)$, $A_n x = n(x_2 - x_1, \ldots, x_n - x_{n-1}, x_1 - x_n) - x$.

2. APPROXIMATION OF THE SPECTRUM FROM ABOVE

We adhere to the notations of the previous section. However we denote the norm on F_n by $\|\cdot\|$ as usual.

In order to obtain the strongest possible results we have to refine the notion of the ε-pseudospectrum of an operator $(A, \mathscr{D}(A))$ defined on the Banach space E. For it turns out that the part $\sigma(A) \setminus \sigma_a(A)$ cannot always be approximated in the general case.

Set $\alpha(A) := \inf\{\|Ax\| : x \in \mathscr{D}(A), \|x\| = 1\}$. Whenever E_0 is a core of A then $\alpha(A) = \inf\{\|Ax\| : x \in E_0, \|x\| = 1\}$ holds. Using this quantity we obtain $\sigma_a(A) = \{\lambda \in \mathbb{C} : \alpha(\lambda - A) = 0\}$. So for $0 \leqslant \varepsilon$ we define the ε-approximate spectrum $\sigma_{\varepsilon, a}$ by

$$\sigma_{\varepsilon, a}(A) = \{\lambda \in \mathbb{C} : \alpha(\lambda - A) \leqslant \varepsilon\}$$

(cf. [9] where the strict inequality $< \varepsilon$ is used). In particular $\sigma_{0,a}(A) = \sigma_a(A)$ as well as $\sigma_{\varepsilon, a}(A) \subset \sigma_\varepsilon(A)$ holds.

LEMMA 2.1. *$\sigma_{\varepsilon, a}(A)$ is always closed.*

Proof. Let $\lambda \notin \sigma_{\varepsilon, a}(A)$. Then $\alpha(\lambda - A) =: \delta > \varepsilon$. Now let $\mu \in \mathbb{C}$ satisfy $|\lambda - \mu| < \delta - \varepsilon$. Then

$$\begin{aligned} \alpha(\mu - A) &= \inf\{\|(\mu - A)\, x\| : \|x\| = 1, x \in \mathscr{D}(A)\} \\ &\geqslant \inf\{|\, \|(\lambda - A)\, x\| - |\lambda - \mu|\, \|x\|\, | : \|x\| = 1, x \in \mathscr{D}(A)\} > \varepsilon. \end{aligned}$$

So the complement of $\sigma_{\varepsilon, a}(A)$ is open. ∎

As for $\varepsilon = 0$ also for $\varepsilon > 0$ it may happpen that $\sigma_{\varepsilon, a}(A) \neq \sigma_\varepsilon(A)$ holds as the following example shows.

EXAMPLE. Let $E = \ell^2(\mathbb{N})$ and let S be the right shift on E given by

$$(Sf)(k) = \begin{cases} 0 & k = 1 \\ f(k-1) & k \geqslant 2. \end{cases}$$

Since S is an isometry on E, it is easily checked that $\alpha(\lambda - S) \geqslant 1 - |\lambda|$ holds for all λ with $0 \leqslant |\lambda| \leqslant 1$. Moreover $\sigma_a(S) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and for $0 < \varepsilon < 1$ we obtain $\sigma_\varepsilon(S) \setminus \sigma_{\varepsilon, a}(S) \supset \{\lambda : |\lambda| < 1 - \varepsilon\}$, the latter set being a subset of the residual spectrum $\sigma_{res}(S) = \{\lambda \in \sigma(S) : \alpha(\lambda - S) > 0\}$ of S.

The main result of this section is a generalization of [16, Theorem 2.2]. There we required A to be continuous. Moreover we assumed that the operators P_n of the approximation scheme are continuous. Finally the conclusion was somewhat weaker than that one of the following theorem.

THEOREM 2.2. *Let $(E, E_1, (F_n), (P_n))$ be a fixed discrete approximation scheme. Moreover, let the sequence (A_n) of densely defined linear operators $(A_n, \mathscr{D}(A_n))$ on the Banach space F_n approximate discretely the closed densely defined linear operator $(A, \mathscr{D}(A))$ on E. Then for every pair $(\varepsilon_1, \varepsilon_2)$ of real numbers with $0 \leqslant \varepsilon_1 < \varepsilon_2$*

$$\sigma_{\varepsilon_1, a}(A) \subset \liminf_n \sigma_{\varepsilon_2, a}(A_n) = \bigcup_{n=1}^{\infty} \bigcap_{k \geqslant n} \sigma_{\varepsilon_2, a}(A_k)$$

holds.

Proof. For the sake of convenience we set $\varepsilon_1 = \eta$, $\varepsilon_2 = \varepsilon$. Let $\lambda \in \sigma_{\eta, a}(A)$ be arbitrary. Choose $\beta > 0$ such that $\varepsilon - \eta - 2\beta > 0$ and set $\gamma = 1 - \frac{\eta + 2\beta}{\varepsilon}$. By hypothesis there exists a core E_0 of A with $A(E_0) \subset E_1$. Then there exists $x \in E_0$ of norm 1 such that $\|(\lambda - A) x\| < \eta + \beta/2$. Again by hypothesis there exists n_0 such that for all $n \geqslant n_0$

$$| \|P_n x\| - \|x\| | < \gamma.$$

$$\|P_n A x - A_n P_n x\| < \beta.$$

$$\|P_n (\lambda - A) x\| < \eta + \beta.$$

Then for all $n \geqslant n_0$ we obtain

$$\begin{aligned} \|\lambda P_n x - A_n P_n x\| &\leqslant \|\lambda P_n x - P_n A x\| + \|P_n A x - A_n P_n x\| \\ &= \|P_n (\lambda - A) x\| + \|P_n A x - A_n P_n x\| < \eta + 2\beta. \end{aligned}$$

Now $\|x\| = 1$ implies $1 - \gamma < \|P_n x\| < 1 + \gamma$. Dividing the inequalities above by $\|P_n x\|$ we get $\|(\lambda - A_n)(P_n x / \|P_n x\|)\| \leqslant \varepsilon$ which implies $\lambda \in \sigma_{\varepsilon, a}(A_n)$ for all $n \geqslant n_0$. ∎

In general $\sigma_a(A) \not\subset \liminf_n(\sigma_a(A_n))$ as Example 5 of the introduction shows. The following example is much easier but perhaps a little more artificial.

EXAMPLE. Let $E = \ell^2(\mathbb{N})$ and let T be the left shift given by $(Tf)(k) = f(k+1)$. Let

$$(A_n f)(k) = \begin{cases} f(k+1) & k \leqslant n-1 \\ 0 & \text{else.} \end{cases}$$

Then (A_n) converges pointwise to T, but $\sigma(A_n) = \{0\}$. Nevertheless

$$\{\lambda : |\lambda| \leqslant 1\} = \sigma_a(T) = \sigma(T) \subset \liminf_n \sigma_{\varepsilon, a}(A_n) \qquad \text{for all} \quad \varepsilon > 0.$$

In the special situation when $\|(\lambda - A_n)^{-1}\| \leqslant M \operatorname{dist}(\lambda, \sigma(A_n))^{-1}$ holds for all n and a suitable $M > 0$ we obtain the following stronger result:

THEOREM 2.3. *Assume in addition to the hypotheses of Theorem 2.2 that there exists $M > 0$ such that $\|(\lambda - A_n)^{-1}\| \leqslant M \operatorname{dist}(\lambda, \sigma(A_n))^{-1}$ holds for all $\lambda \in \rho(A_n)$ and for all n. Then for every compact subset $K \subset \mathbb{C}$*

$$\lim_{n \to \infty} \operatorname{dist}(\sigma_a(A) \cap K, \sigma(A_n)) = 0.$$

Proof. Assume that the assertion fails. Then there exists a $\delta > 0$, a compact set $K \subset \mathbb{C}$ and a sequence (λ_{n_k}) in $\sigma_a(A) \cap K$ with $\operatorname{dist}(\lambda_{n_k}, \sigma(A_{n_k})) \geqslant \delta > 0$ for all k. Since K is compact there exists a subsequence $(\lambda_{n'_k})$ converging to a point $z \in \sigma_a(A) \cap K$ since $\sigma_a(A)$ is closed. By applying Theorem 2.2 with $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon = \frac{\delta}{2(1+2M)}$ we obtain n_0 such that $z \in \sigma_\varepsilon(A_n)$ for all $n \geqslant n_0$. Moreover there exists k_0 with $|z - \lambda_{n'_k}| < \varepsilon$ for all $k \geqslant k_0$. This implies $z \notin \sigma(A_{n'_k})$. But then $M \operatorname{dist}(z, \sigma(A_{n'_k}))^{-1} \geqslant \|(z - A_{n'_k})^{-1}\| \geqslant \frac{1}{\varepsilon}$ yields $M\varepsilon \geqslant \operatorname{dist}(z, \sigma(A_{n'_k}))$. Hence there exists $\mu_{n'_k} \in \sigma(A_{n'_k})$ with $|z - \mu_{n'_k}| < 2\varepsilon M$ for all $k \geqslant k_1 \geqslant k_0$. This in turn implies

$$|\lambda_{n'_k} - \mu_{n'_k}| \leqslant |\lambda_{n'_k} - z| + |z - \mu_{n'_k}| < (1+2M)\,\varepsilon = \delta/2$$

for $k \geqslant k_1$, a contradiction to $\operatorname{dist}(\lambda_{n'_k}, \sigma(A_{n'_k})) \geqslant \delta$. ∎

Here are some applications:

EXAMPLES (1) For S the right shift on $E = \ell^2(\mathbb{N})$ we have $\sigma_a(S) = \{\lambda : |\lambda| = 1\}$ (cf. the example after Lemma 2.1). Set $F_n = \mathbb{C}^n$ and $P_n : E \to F_n$, $f \to (f(1), \ldots, f(n))$. Let $A_n(x) = (x_n, x_1, \ldots, x_{n-1})$. Then A_n is unitary hence normal, and thus we can apply Theorem 2.3 with $M = 1$ to obtain

$\lim \operatorname{dist}(\sigma_a(S), \sigma(A_n)) = 0$. However the residual spectrum $\sigma(S) \setminus \sigma_a(S)$ cannot be approximated by $\sigma(A_n)$.

(2) Let E, E_1, P_n, F_n, A, and A_n be as in Example 4 of the Introduction. Then $\sigma_a(A) = \sigma(A) = 2\pi i \mathbb{Z}$ and the approximating operators A_n are normal with $\sigma(A_n) = \{n(\exp(2\pi ik/n) - 1) : 0 \leqslant k \leqslant n-1\}$. We obtain $2\pi i \mathbb{Z} \subset \liminf_n \sigma_\varepsilon(A_n)$. Moreover $\lim \operatorname{dist}(\sigma(A) \cap K, \sigma(A_n)) = 0$ as follows also directly from $2\pi ik = \lim_{n \to \infty} n(\exp(2\pi ik/n) - 1)$ for each fixed k.

(3) Let E, E_1, P_n, F_n, A, and A_n be as in Example 5 of the Introduction. Then $A_n = n(N_n - Q_n)$ where Q_n is the projection onto the first $(n-1)$ coordinates and $N_n^n = 0$. Hence $\sigma(A_n) = \{-n, 0\}$. Theorem 2.3 does not apply since there is no M satisfying the hypothesis of this theorem. However $2\pi i \mathbb{Z} \subset \liminf \sigma_\varepsilon(A_n)$ as follows from Theorem 2.2. It can also easily be deduced directly by verifying $\|(2\pi\, ik - A_n)\, e_{k,n}\| = O(n^{-1/2})$ where $e_{k,n} = (\exp(\frac{2\pi\, ikl}{n}))_{l=1,\ldots,n}$ for fixed $k \in \mathbb{Z}$.

Remark 2.4. Let us point out that in special cases there are more precise results even if the approximants are nonnormal operators. As an example we give the following theorem due to Böttcher [2]: *Let a be a piecewise continuous symbol on the unit circle $\mathbb{T}$ and consider the approximation of the associated Toeplitz operator $T(a)$ on $\ell^2(\mathbb{N})$ given by $(T(a)(x))_n = \sum_0^n a_{n-k} x_k$ where $(a_n)_{n \in \mathbb{Z}}$ is the sequence of Fourier coefficients of a. Let T_n be the nth section operator of $T(a)$. Then for all $\varepsilon > 0$*

$$\sigma_\varepsilon(T(a)) = \lim_{n \to \infty} \sigma_\varepsilon(T_n)$$

holds where the limit is taken with respect to the Hausdorff metric $d_H(X, Y) = \max(\operatorname{dist}(X, Y), \operatorname{dist}(Y, X))$ on the space of all compact subsets of $\mathbb{C}$.

3. APPROXIMATION OF THE SPECTRUM FROM BELOW

In this section we turn to the problem of when

$$\bigcap_{\varepsilon > 0} \bigcap_{n \geqslant 1} \bigcup_{k \geqslant n} \sigma_{\varepsilon, a}(A_k) \subset \sigma_a(A)$$

holds where the sequence (A_n) approximates A as before. Let $(r_n)_n$ be an arbitrary sequence of positive real numbers converging to 0. Then

$$\bigcap_{\varepsilon > 0} \bigcap_{n \geqslant 1} \bigcup_{k \geqslant n} \sigma_{\varepsilon, a}(A_k) = \bigcap_{n \geqslant 1} \bigcup_{k \geqslant n} \sigma_{r_n, a}(A_k) \tag{1}$$

is easily seen to hold. So in the following proofs we prefer the right-hand side description of this set because it contains only two set theoretical operations.

In order to get our results we have to make use of the theory of ultraproducts of Banach spaces (see [12] for details). Let $\mathcal{G} = (G_n)_n$ be a sequence of Banach spaces. Then $\ell_\infty(\mathcal{G})$ is the subspace of all norm bounded sequences of the Cartesian product $\prod_n G_n$. Equipped with the norm $\|(x_n)\| := \sup\{\|x_n\| : n \in \mathbb{N}\}$ $\ell_\infty(\mathcal{G})$ is a Banach space. Now let $\mathcal{U}$ be an arbitrary free ultrafilter on $\mathbb{N}$. Then $c_{0,\mathcal{U}}(\mathcal{G}) = \{(x_n)_n \in \ell_\infty(\mathcal{G}) : \lim_{\mathcal{U}} \|x_n\| = 0\}$ is a closed subspace of $\ell_\infty(\mathcal{G})$. The quotient space is called the ultraproduct $\mathcal{G}_{\mathcal{U}}$ of $\mathcal{G}$ with respect to $\mathcal{U}$. If $\xi = (x_n)_n$ is in $\ell_\infty(\mathcal{G})$ then

$$\|\hat{\xi}\| = \|\xi + c_{0,\mathcal{U}}(\mathcal{G})\| = \lim_{\mathcal{U}} \|x_n\| \qquad \text{holds.}$$

Let $U \in \mathcal{U}$ be arbitrary. Then $\hat{\xi}$ is determined already by the subsequence $(x_{n'})_{n' \in U}$, a fact which we will use tacitely in the sequel.

Now let $(T_n)_n$ be a uniformly bounded sequence of operators $T_n \in \mathcal{L}(G_n)$. Then by $\tilde{T}\xi = (T_n x_n)_n$ there is defined a bounded linear operator $\tilde{T}$ on $\ell_\infty(\mathcal{G})$ for which $c_{0,\mathcal{U}}(\mathcal{G})$ is an invariant subspace. The operator $\hat{T}$ on the quotient space $\mathcal{G}_{\mathcal{U}}$ is called the ultraproduct of (T_n) with respect to $\mathcal{U}$. Let $U \in \mathcal{U}$ be arbitrary. Similarly to the fact in the previous paragraph $\hat{T}$ does not depend on indices not in U. In particular $\hat{T}$ does not depend on the first n_0 operators. More generally if the sequence is only defined for $n \in U$ then we may fill up it with arbitrary bounded operators $(T_n)_{n \notin \mathcal{U}}$ obtaining always the same operator $\hat{T}$.

Let $(E, E_1, (F_n), (P_n))$ be a given approximation scheme. Let $\mathcal{F}_{\mathcal{U}}$ be the ultraproduct of $(F_n)_n$ with respect to a given free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Since $\lim \|P_n y\| = \|y\|$ holds by hypothesis for all $y \in E_1$ we obtain an isometry $V_{\mathcal{U}} : E_1 \to \mathcal{F}_{\mathcal{U}}$ by $V_{\mathcal{U}}(y) = \widehat{(P_n y)_n}$. Since E_1 is dense in E $V_{\mathcal{U}}$ can be uniquely extended to an equally denoted isometry on E.

Let the closed densely defined operator $(A, \mathcal{D}(A))$ on E be discretely approximated by the sequence $(A_n)_n$ of bounded linear operators A_n on F_n. If this sequence is uniformly bounded then we obtain easily

$$\hat{A} V_{\mathcal{U}}|_{E_0} = V_{\mathcal{U}} A|_{E_0}.$$

PROPOSITION 3.1. *Let $(E, E_1, (F_n), (P_n))$ be a given approximation scheme. Let $(A, \mathcal{D}(A))$ be discretely approximated by the sequence $((A_n, \mathcal{D}(A_n)))_n$. Moreover assume that A as well as all A_n are surjective, and that*

$$\liminf_{n \to \infty} \alpha(A_n) > 0.$$

Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ A_n is bijective. Moreover A_n^{-1} is bounded, the sequence $(A_n^{-1})_{n \geqslant n_0}$ is uniformly bounded, and A^{-1} exists and is discretely approximated by $(A_n^{-1})_{n \geqslant n_0}$. Finally

$$V_{\mathscr{U}} A^{-1} = \widehat{(A_n^{-1})}\, V_{\mathscr{U}}$$

holds for every free ultrafilter $\mathscr{U}$ on $\mathbb{N}$.

Proof. There exists $\eta > 0$ and $n_0 \in \mathbb{N}$ such that $\alpha(A_n) \geqslant \eta$ for all $n \geqslant n_0$. Since all A_n are surjective, A_n is bijective for $n \geqslant n_0$ and $\|A_n^{-1}\| \leqslant \frac{1}{\eta}$ holds for all these n.

Let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$. Then the operator $B = \widehat{(A_n^{-1})}_{n \geqslant n_0}$ is well defined on $\mathscr{F}_{\mathscr{U}}$ and $\|B\| \leqslant \frac{1}{\eta}$. Assume now that there exists $x_0 \in E_0$, $\|x_0\| = 1$ with $\|Ax_0\| < \delta = \min(1, \eta)/2$. By hypothesis the following assertions hold:

$$\|V_{\mathscr{U}}(x_0)\| = 1,$$
$$\|V_{\mathscr{U}}(Ax_0)\| = \|Ax_0\| < \delta,$$
$$V_{\mathscr{U}}(Ax_0) = (A_n P_n x_0)_n^{\wedge}.$$

But then $\|(A_n P_n x_0)_n^{\wedge}\| = \|V_{\mathscr{U}}(Ax_0)\| < \delta$. This in turn implies

$$\|((A_n P_n x_0)/\|P_n x_0\|)_n^{\wedge}\| < 2\delta,$$

a contradiction to $\alpha(A_n) \geqslant \eta \geqslant 2\delta$ for all $n \geqslant n_0$. So $\alpha(A) \geqslant \delta$, since E_0 is a core of A. Since A is surjective, A^{-1} exists and is bounded with norm $\leqslant 1/\delta$. Moreover $A(E_0)$ is dense in E. For if $y \in E$ is arbitrary then $(A^{-1}y, y)$ is contained in the graph $G(A)$ of A. But since E_0 is a core $G(A) = \overline{G(A|_{E_0})}$ and the assertion follows.

Now let $z \in A(E_0)$ be arbitrary and set $y = A^{-1}z$. Then $V_{\mathscr{U}} z = V_{\mathscr{U}} A y = (A_n P_n y)_n^{\wedge}$. This implies

$$BV_{\mathscr{U}} z = (A_n^{-1}(A_n P_n y))_n^{\wedge} = V_{\mathscr{U}} A^{-1} z$$

for all $z \in A(E_0)$. Since this latter space is dense in E the final equality follows from the continuity of B, $V_{\mathscr{U}}$ and A^{-1}.

Finally in order to prove that A^{-1} is discretely approximated by the sequence $(A_n^{-1})_n$ we have to specify a core $E_2 \subset E_1$ of A^{-1} with $A^{-1}(E_2) \subset E_1$. But since $A(E_0)$ is dense it may serve as such a core. ∎

The following conditions on the approximation scheme $(E, E_1, (F_n), (P_n))$ are designed for applications to the approximation of operators on infinite-dimensional manifolds (see the example following the next theorem):

Assume that for every $m < n$ there exists a linear isometric embedding $S_{n,m}$ from F_m into F_n. Moreover let the sequence $((A_n, \mathscr{D}(A_n)))$ approximate the closed densely defined operator $(A, \mathscr{D}(A))$. To every n let G_n be a core of A_n and as before let E_0 be a core of A with $A(E_0) \subset E_1$.

THEOREM 3.2. *In addition to the assumptions made in the previous paragraph let the following condition be satisfied: For all $k \in \mathbb{N}$ and for all $z \in G_k$ there exists $y \in E_0$ and an unbounded sequence (t_n) such that*

$$P_{t_n} y = S_{t_n, k} z \qquad \textit{for all } t_n$$

as well as

$$\lim_{n \to \infty} \|A_{t_n} S_{t_n, k} z - S_{t_n, k} A_k z\| = 0.$$

Then

$$\bigcap_{\varepsilon > 0} \bigcap_{n \geqslant 1} \bigcup_{k \geqslant n} \sigma_{\varepsilon, a}(A_k) \subset \sigma_a(A).$$

Proof. We use Eq. (1) with $r_n = 1/n$. Assume that the assertion fails. Then there exists $\lambda \in \bigcap_{n \in \mathbb{N}} \bigcup_{k \geqslant n} \sigma_{r_n, a}(A_k)$ with $\alpha(\lambda - A) = \delta > 0$. By hypothesis there exists a sequence $(k_n)_n$ with $k_n \geqslant n$ and $\alpha(\lambda - A_{k_n}) \leqslant r_n$. To each n there exists $x_{k_n} \in G_{k_n}$ of norm 1 such that $\|(\lambda - A_{k_n}) x_{k_n}\| \leqslant 2r_n$. Choose n_0 such that $r_n < \delta/3$ for all $n \geqslant n_0$. Fix $n_1 \geqslant n_0$ and choose an ultrafilter $\mathscr{U}$ with $\{t_n : n \in \mathbb{N}\} \in \mathscr{U}$ where $(t_n)_n$ is the sequence for the element $z = x_{k_{n_1}}$ as required in the hypothesis. By assumption there exists an element y in E_0 with

$$P_{t_n} y = S_{t_n, k_{n_1}}(z) \qquad \text{for all} \quad n \in \mathbb{N}.$$

Again by hypothesis

$$\lim_{n \to \infty} \|S_{t_n, k_{n_1}}(A_{k_{n_1}}(z)) - A_{t_n}(S_{t_n, k_{n_1}}(z))\| = 0$$

holds. Since $S_{t_n, k_{n_1}}(z) = P_{t_n} y$ we obtain

$$\begin{aligned} (\widehat{S_{n, k_{n_1}} A_{k_{n_1}} z}) &= (\widehat{A_n S_{n, k_{n_1}} z}) \\ &= (\widehat{A_n P_n y}) = V_{\mathscr{U}} A y. \end{aligned}$$

Notice that we have tacitly made use of the fact that elements in the ultraproduct do not depend on values x_n, A_n, etc. for indices n not contained in $\{t_m : m \in \mathbb{N}\}$.

Because $(E, E_1, (F_n), (P_n))$ is an approximation scheme we have

$$\|V_{\mathscr{U}} y\| = \|y\| = \|S_{t_n, k_{n_1}} z\| = \|z\| = 1.$$

Since $S_{n,k_{n_1}}$ are isometries we obtain

$$2r_{n_1} \geqslant \|(\lambda - A_{k_{n_1}})\, z\| = \|(\lambda S_{t_n,k_{n_1}} z - S_{t_n,k_{n_1}} A_{k_{n_1}})\, z\|$$
$$\geqslant \big| \|\lambda P_{t_n} y - A_{t_n} S_{t_n,k_{n_1}} z\| - \|A_{t_n} S_{t_n,k_{n_1}} z - S_{t_n,k_{n_1}} A_{k_{n_1}} z\| \big| \qquad \text{for all } n.$$

This inequality yields

$$2r_{n_1} \geqslant \|\lambda V_{\mathscr{U}}\, y - V_{\mathscr{U}} A y\| = \|V_{\mathscr{U}}(\lambda - A)\, y\| \geqslant \alpha(\lambda - A) = \delta > 3r_{n_1},$$

a contradiction. ▌

Example (cf. [8] as well as Example (3) in Section 1). Let H be a separable, infinite dimensional Hilbert space over $\mathbb{R}$ with orthonormal basis (e_n). Let E be the space of uniformly continuous complex valued bounded functions on H equipped with the supremum norm. For each n let Q_n denote the orthogonal projection of H onto the span H_n of $e_1, ..., e_n$ and set $F_n = \{f \in E : f = f \circ Q_n\}$. Then $P_n : E \to F_n$, $f \mapsto f \circ Q_n$ is a projection of norm 1, and moreover F_n is isometrically isomorphic to the space of all bounded uniformly continuous functions on $\mathbb{R}^n$. So we identify these two spaces. Then the isometry $S_{n,m}$ whose existence is required in Theorem 3.2 is nothing else than the inclusion mapping.

Let (λ_n) be a positive summable sequence and set $A_n = \sum_{k=1}^{n} \lambda_k (\partial^2/\partial x_k^2)$. Then (A_n) approximates the infinite dimensional Laplacian $\sum_{k=1}^{\infty} \lambda_k (\partial^2/\partial x_k^2)$. In order to apply Theorem 3.2 we set $\mathscr{D}(A_n) = G_n$, $E_0 = \bigcup \mathscr{D}(A_n)$ and $E_1 = E$. Moreover if $z \in G_k$ then we take $(t_n)_n = (n)_{n \geqslant k}$ and $y = z$. Then all the assumptions made in Theorem 3.2 are satisfied. We show that

$$\bigcap_n \bigcup_{k \geqslant n} \sigma_{1/n,\, a}(A_k) = \{\lambda : \Re(\lambda) \leqslant 0\}.$$

Our proof borrows an idea from the original proof of the main result of [8] which however does not use the notion of ε-pseudospectrum. To this end let $\lambda \in \mathbb{C}$ with $\Re(\lambda) =: \gamma < 0$ be arbitrary. Then the function

$$g_m(x) = \exp\left(\frac{\lambda}{2m} \sum_{j=1}^{m} \frac{x_j^2}{\lambda_j} \right)$$

is of norm 1 and a short calculation shows

$$(\lambda - A_m)\, g_m(x) = -\frac{\lambda^2}{m^2}\, g_m(x) \cdot \sum_{j=1}^{m} \frac{x_j^2}{\lambda_j}.$$

The inequality

$$t \exp\left(\frac{\gamma}{2m}\, t \right) \leqslant \frac{2m}{-\gamma\, e}$$

for all $t \geqslant 0$ which is proved by elementary calculus shows

$$\alpha(\lambda - A_m) \leqslant \|(\lambda - A_m)\, g_m\| \leqslant \frac{2\,|\lambda|^2}{|\Re(\lambda)|\, em}$$

which in turn proves that $\lambda \in \sigma_{1/n,\,a}(A_m)$ for $m > 2n\,|\lambda|^2/|\Re(\lambda)|\, e$. So Theorem 3.2 implies

$$\{\lambda\colon \Re(\lambda) < 0\} \subset \bigcap_{n \geqslant 1} \bigcup_{k \geqslant n} \sigma_{1/n,\,a}(A_k) \subset \sigma_a(A).$$

Finally we apply Theorem 2.2: A_n is known to be the generator of a contraction semigroup. The Hille–Yosida Theorem (see [10, p. 8]) implies $\|(\lambda - A_n)^{-1}\| \leqslant (\Re(\lambda))^{-1}$ for all λ with $\Re(\lambda) > 0$ independently of n. So $\lambda \in \rho_\varepsilon(A_n)$ for $\Re(\lambda) > \varepsilon$. Theorem 2.2 then implies $\sigma_a(A) \subset \{\lambda\colon \Re(\lambda) \leqslant 0\}$.

Our final results are concerned with discretely compact approximation. A sequence $(x_n) \in \prod F_n$ is called *discretely compact* (or d-compact for short) if for every $\varepsilon > 0$ there exists a finite set $Y(\varepsilon) \subset E_1$ depending on ε such that

$$\limsup_n \operatorname{dist}(x_n, P_n(Y(\varepsilon))) < \varepsilon.$$

Discretely compact sequences can be described as follows:

LEMMA 3.3. *Let $(x_n)_n$ be a discretely compact sequence. Then to every free ultrafilter $\mathscr{U}$ there exists $y \in E$ such that $V_{\mathscr{U}}\, y = \widehat{(x_n)_n}$.*

Proof. Let $\mathscr{U}$ be a fixed free ultrafilter. By hypothesis to every $r \in \mathbb{N}$ there exists a finite set $Y(r)$ in E_1 depending on r such that $\limsup_n \operatorname{dist}(x_n, P_n(Y(r))) < 2^{-r}$. Since $\mathscr{U}$ is an ultrafilter and $Y(r)$ is finite there exists some $y_r \in Y(r)$ such that $\{n\colon \|x_n - P_n\, y_r\| < 2^{-r}\} \in \mathscr{U}$. This in turn implies $\|(x_n)_n^{\wedge} - V_{\mathscr{U}}\, y_r\| < 2^{-r}$. Let $p \in \mathbb{N}$ be arbitrary. Since $V_{\mathscr{U}}$ is an isometry we obtain that

$$\|y_{r+p} - y_r\| \leqslant \|V_{\mathscr{U}}(y_{r+p}) - \widehat{(x_n)}\| + \|\widehat{(x_n)} - V_{\mathscr{U}}(y_r)\| < 2^{-r+1},$$

hence (y_r) is a Cauchy sequence. If $y = \lim y_r$ then obviously $V_{\mathscr{U}}\, y = \widehat{(x_n)_n}$. ∎

Let $(A, \mathscr{D}(A))$ be discretely approximated by the sequence $((A_n, \mathscr{D}(A_n)))$. We say that the approximation is *discretely compact* if (A_n) is uniformly bounded and for every bounded sequence (x_n) the sequence $(A_n x_n)$ is d-compact. For examples and for the connection to collectively compact sequences in the sense of [1] see [11, Sect. 7.3]. (A_n) is called *inverse d-compact* if the sequence (x_n) is d-compact whenever $(A_n x_n)$ is bounded.

LEMMA 3.4. *Let (A_n) be inverse d-compact. Then* $\liminf_{n\to\infty} \alpha(A_n) > 0$. *If moreover A and all A_n are surjective, then there exists $n_0 \in \mathbb{N}$ such that all A_n are bijective and the approximation of A^{-1} by $(A_n^{-1})_{n \geqslant n_0}$ is discretely compact.*

Proof. If the assertion fails then to every $k \in \mathbb{N}$ there exists $n_k \geqslant k$ and x_{n_k} with $\|x_{n_k}\| = 1$ and $\|A_{n_k} x_{n_k}\| < 2^{-k}$. Set

$$y_n = \begin{cases} 0, & n \notin \{n_k : k \in \mathbb{N}\} \\ 2^k x_{n_k} & n = n_k \quad \text{for some} \quad k \in \mathbb{N}. \end{cases}$$

Then (y_n) is unbounded hence not d-compact though $(A_n y_n)$ is bounded.

Now let all A_n be surjective. By Proposition 3.1 there exists n_0 such that for all $n \geqslant n_0$ A_n is bijective. Moreover A_n^{-1} is bounded and the sequence $(A_n^{-1})_{n \geqslant n_0}$ is uniformly bounded and approximates A^{-1} discretely. Finally let $(x_n)_n$ be bounded and set $y_n = A_n^{-1} x_n$. Then $(A_n(y_n))_{n \geqslant n_0}$ is bounded hence $(y_n)_n$ is discretely compact by hypothesis. So the assertion follows. ∎

Let now $T \in \mathscr{L}(E)$ with $T(E) \subset E_1$ and let (T_n) be a uniformly bounded sequence of operators $T_n \in \mathscr{L}(F_n)$. Since E_1 may serve as a core of T the discrete approximation of T by (T_n) can be defined unambiguously. Recall that we denote the set of eigenvalues of the operator T by $\sigma_p(T)$.

PROPOSITION 3.5. *Let $T, (T_n)$ be as above. Assume that (T_n) approximates T discretely and moreover that (T_n) is discretely compact. Then*

$$\bigcap_{\varepsilon > 0} \bigcap_{n \geqslant 1} \bigcup_{k \geqslant n} \sigma_{\varepsilon, a}(T_k) \subset \sigma_p(T) \cup \{0\}.$$

Proof. Again we use Eq. (1) with $r_n = 1/n$. Let $0 \neq \lambda \in \bigcap_{n \in \mathbb{N}} \bigcup_{k \geqslant n} \sigma_{r_n, a}(T_k)$ be arbitrary. Then there exists a sequence $(k_n)_n$ with $k_n \geqslant n$ and $\lambda \in \sigma_{r_n, a}(T_{k_n})$. This in turn implies the existence of a normalized vector $x_{k_n} \in F_{k_n}$ with $\|(\lambda - T_{k_n}) x_{k_n}\| \leqslant 2r_n$. Set $x_l = 0$ for $l \notin \{k_n : n \in \mathbb{N}\}$. Let $\mathscr{U}$ be an ultrafilter containing $\{k_n : n \in \mathbb{N}\}$. Then for $\xi = (x_n)_n$ we obtain $\|\hat{\xi}\| = 1$ as well as $0 \neq \lambda\hat{\xi} = \hat{T}\hat{\xi}$. Since by hypothesis the sequence $(T_n x_n)_n$ is discretely compact by Lemma 3.3 there exists $y \in E$ with $\hat{T}\hat{\xi} = (T_n x_n)_n^{\wedge} = V_{\mathscr{U}} y$ which in turn gives $\lambda\hat{\xi} = V_{\mathscr{U}} y$. Alltogether we obtain

$$\lambda V_{\mathscr{U}} y = \lambda \hat{T}\hat{\xi} = \hat{T}(\lambda\hat{\xi}) = \hat{T} V_{\mathscr{U}} y = V_{\mathscr{U}} T y$$

where the last equation holds since $(T_n)_n$ approximates T. Because $V_{\mathscr{U}}$ is an isometry this latter equation yields $\lambda y = T y$. ∎

In most cases the spaces F_n are finite-dimensional so that $\sigma_a(T_n)=\sigma_p(T_n)$ holds in the following proposition.

PROPOSITION 3.6. *Under the assumptions of Proposition* 3.5

$$\lim_{n\to\infty} \operatorname{dist}(\sigma_a(T_n), \sigma_p(T) \cup \{0\}) = 0$$

holds.

Proof. Assume that the assertion does not hold. Then there exists $\delta>0$ and a sequence $(k_n)_n$ with $k_n \geqslant n$ and moreover to every n a $\lambda_{k_n} \in \sigma_a(T_{k_n})$ such that $\inf\{|\lambda_{k_n} - \nu| : \nu \in \sigma_p(T) \cup \{0\}\} \geqslant \delta$. Since $(T_n)_n$ is uniformly bounded (λ_{k_n}) is bounded. Set $\lambda_l = 0$ for $l \notin \{k_n : n \in \mathbb{N}\}$ and let $\mathscr{U}$ be a free ultrafilter containing $\{k_n : n \in \mathbb{N}\}$. Since $(\lambda_n)_n$ is bounded it converges along $\mathscr{U}$. Set $\mu = \lim_{\mathscr{U}} \lambda_n$. Then $|\mu| \geqslant \delta > 0$. For every n we choose a normalized vector $x_{k_n} \in F_{k_n}$ with $\|(\lambda_{k_n} - T_{k_n})\, x_{k_n}\| < 2^{-n}$. We set $x_l = 0$ for $l \notin \{k_n : n \in \mathbb{N}\}$. Let $\xi = (x_n)_n$. Then we obtain $\mu\hat{\xi} = \hat{T}\hat{\xi}$. By Lemma 3.3 there exists $y \in E$ with $\hat{T}\hat{\xi} = V_{\mathscr{U}}\, y$. As in the proof of the preceding proposition we obtain $\mu \in \sigma_p(T)$, a contradiction to $\operatorname{dist}(\lambda_{k_n}, \sigma_p(T) \cup \{0\}) \geqslant \delta$. ∎

Our main result is now an immediate consequence of this proposition:

THEOREM 3.7. *Let $((A_n, \mathscr{D}(A_n)))_n$ be an inverse d-compact sequence of operators approximating discretely the closed densely defined operator $(A, \mathscr{D}(A))$ in E. Assume that A as well as all A_n are surjective and that $A(E_0) \subset E_1$ is dense in E. Then for every compact set $K \neq \varnothing$ in $\mathbb{C}$*

$$\lim_{n} \operatorname{dist}(\sigma_a(A_n) \cap K, \sigma_p(A)) = 0.$$

Proof. By Lemma 3.4 the sequence (A_n) satisfies the hypotheses of Proposition 3.1 and moreover the inverse operators A_n^{-1} which exist from some n_0 on form a discretely compact approximation of A^{-1}. Hence by Proposition 3.6

$$\lim_{n\to\infty} \operatorname{dist}(\sigma_a(A_n^{-1}), \sigma_p(A^{-1}) \cup \{0\}) = 0 \tag{2}$$

holds. By the spectral mapping theorem $\sigma_a(A_n) = \{\frac{1}{\lambda} : \lambda \in \sigma_a(A_n^{-1})\}$ and $\sigma_p(A) = \{\frac{1}{\lambda} : \lambda \in \sigma_p(A^{-1})\}$. If the assertion does not hold there exists a sequence $(\lambda_{n_k})_k$ with $\lambda_{n_k} \in \sigma_a(A_{n_k}) \cap K$ and

$$\operatorname{dist}(\lambda_{n_k}, \sigma_p(A)) \geqslant \delta > 0 \tag{3}$$

for all k. Since K is compact w.l.o.g. we assume that $(\lambda_{n_k})_k$ converges to some $\mu \in K$. Because $1/|\lambda_{n_k}| \leqslant \|A_{n_k}^{-1}\|$ and this latter sequence is bounded

$\mu \neq 0$ holds. Since $(1/\lambda_{n_k})_k$ converges to $1/\mu$ it follows by Eq. (2) that $1/\mu \in \sigma_p(A^{-1})$ hence $\mu \in \sigma_p(A)$, a contradiction to the inequality (3). ∎

EXAMPLES. (1) Obviously Theorem 2.3 as well as Theorem 3.7 applies to Example (6) of the Introduction. So we get in this case

$$\lim_{n \to \infty} \operatorname{dist}(\sigma_a(A) \cap K, \sigma(A_n)) = 0 = \lim_{n \to \infty} \operatorname{dist}(\sigma_a(A_n) \cap K, \sigma_p(A)),$$

from which we deduce $\sigma_a(A) = \sigma_p(A)$ as well as $\lim_{n \to \infty} (\sigma(A_n) \cap K) = \sigma_p(A) \cap K$ with respect to the Hausdorff metric on the set of compact subsets of $\mathbb{C}$.

(2) Let $d > 0$ be given and let E be the space of all continuous functions on the compact interval $[0, d]$ vanishing at 0. Let $\mathscr{D}(A)$ be the subspace of all continuously differentiable functions in E, for which $f'(0) = 0$ holds and set $Af = -f'$. Let $F_n = \mathbb{C}^n$ be equipped with the maximum norm and set $P_n: E \to F_n$, $f \mapsto (f(d/n), f(2d/n), \ldots, f(nd/n))$. Choose $A_n = \frac{n}{d}(-I + T_n)$ with

$$T_n e_k = \begin{cases} e_{k+1}, & k < n \\ 0, & k = n, \end{cases}$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of F_n. It is not hard to prove that (A_n) is inverse d-compact, and $\sigma(A_n) = -n/d$. Indeed in this example Theorem 3.7 gives no information. We obtain $\sigma(A) = \varnothing$ immediately from $((\lambda - A)^{-1} g)(x) = \exp(-\lambda x) \int_0^x \exp(\lambda s)\, g(s)\, ds$ (cf. also [14, Example 3]).

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